On Zero Divisor Graphs of Multiplicative Lattices

Minor Research Project

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Introduction

In recent years much attention has been given to the study of zero-divisor graphs of algebraic structures and ordered structures. The idea of a zero-divisor graph of a commutative ring R with unity was introduced by Beck [8] as follows. Let G be the simple graph whose vertices are the elements of R, with x and y adjacent if $xy = 0$. The graph G is the *zero-divisor graph* of R. The chromatic number of a graph G is denoted by $\chi(G)$. That is, $\chi(G)$ is the minimum number of colors in a coloring of the elements of R such that adjacent elements receive different colors. If this number is not finite, write $\chi(G) = \infty$. A subset C of G is a *clique* if any two distinct vertices of C are adjacent. The *clique number* of a graph G, written $\omega(G)$, is the maximum number of vertices in a clique in G. If the sizes of the cliques are not bounded, then $\omega(G) = \infty$. Always $\chi(G) \geq \omega(G)$. In [8], Beck conjectured that $\chi(G) = \omega(G)$ when G is the zero-divisor graph of a commutative ring with unity, but Anderson and Naseer [6] gave an example of a commutative local ring R with 32 elements for which $\chi(G) > \omega(G)$.

Many researchers such as Anderson et al. [7], F. DeMeyer, T. McKenzie and K. Schneider [11], Maimani, Pournaki and Yassemi [27], Redmond [29], and Samei [30] investigated the interplay between algebraic properties of a structure and its graph-theoretic properties. The zero-divisor graphs of ordered structures were recently studied by Halaš and Jukl [14], Halaš and Länger [15], Joshi [16], Joshi et al. [18, 19, 20, 24, 25, 26], Nimbhorkar et al. [28] etc.

In ring theory, the structure of a ring R is closely related to the behavior of ideals. Hence Behboodi and Rakeei [9, 10] introduced the concept of an annihilating-ideal graph $\mathbb{AG}(R)$ of a commutative ring R with unity, where the vertex set $V(\mathbb{AG}(R))$ is the set of nonzero ideals with nonzero annihilator. That is, a nonzero ideal I belongs to $V(\mathbb{AG}(R))$ if and only if there exists a nonzero ideal J of R such that $IJ = (0)$, and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. In [10], Behboodi and Rakeei raised the following conjecture.

Conjecture 0.1. *For every commutative ring R with unity, $\chi(\mathbb{AG}(R)) = \omega(\mathbb{AG}(R))$.*

It is interesting to observe that the set $Id(R)$ of all ideals of a commutative ring R with unity forms a compactly generated 1-compact modular multiplicative lattice in which the product of two compact elements is compact. Also, the annihilating-ideal graph of a commutative ring R with unity is nothing but the multiplicative zero-divisor graph of the multiplicative lattice of all ideals of R, where the vertex set is the set of nonzero zero-divisors and vertices a and b are adjacent if and only if $ab = 0$. Hence when studying the annihilating-ideal graphs of a commutative ring with unity, a multiplicative lattice becomes an appropriate tool. This motivates us to define and study the multiplicative zero-divisor graph $\widetilde{\Gamma}_I(L)$ of a multiplicative lattice L with respect to an ideal I of L. We say that a multiplicative lattice has the *Beck property* if the chromatic number and clique number of its multiplicative zero-divisor graph with respect to any ideal are equal. It is natural to ask the following question; an affirmative answer to it proves Conjecture 1.1. of Behboodi and Rakeei [10].

Question 0.2. Does the Beck property hold for a given multiplicative lattice?

In this research report, we deal with the basic properties such as connectivity, diameter, girth (gr), clique number (ω), chromatic number (χ) etc. of the multiplicative zero-divisor graph of a multiplicative lattice. This report contains two chapters.

Chapter 1

Beck's Conjecture and multiplicative lattices

*The paper titled "**Beck's Conjecture and multiplicative lattices**" based on the text of this Chapter is published in the journal **Discrete Math.**, **338** (2015), 93-98.*

In this Chapter, we introduce the multiplicative zero-divisor graph of a multiplicative lattice and study Beck-like coloring of such graphs. Further, it is proved that for such graphs, the chromatic number and the clique number need not be equal. On the other hand, if a multiplicative lattice L is reduced, then the chromatic number and the clique number of the multiplicative zero-divisor graph of L are equal, which extends the result of Behboodi and Rakeei [10] and Aalipour et al. [1].

Chapter 2

Diameter and girth of multiplicative zero-divisor graph of multiplicative lattices

In this Chapter, we study the multiplicative zero-divisor graph $\widetilde{\Gamma}(L)$ of a multiplicative lattice L. Under certain conditions, we prove that a reduced multiplicative lattice L having more than two minimal prime elements, $\widetilde{\Gamma}(L)$ contains a cycle and $gr(\widetilde{\Gamma}(L)) = 3$. This essentially settles the conjecture of Behboodi and Rakeei [10]. Further, we have characterized the diameter of $\widetilde{\Gamma}(L)$.

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