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# Beck's conjecture and multiplicative lattices

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ABSTRACT

In this paper, we introduce the multiplicative zero-divisor graph of a multiplicative lattice and study Beck-like coloring of such graphs. Further, it is proved that for such graphs, the chromatic number and the clique number need not be equal. On the other hand, if a multiplicative lattice $L$ is reduced, then the chromatic number and the clique number of the multiplicative zero-divisor graph of $L$ are equal, which extends the result of Behboodi and Rakeei (2011) and Aalipour et al. (2012).



## 1. Introduction

In recent years much attention has been given to the study of zero-divisor graphs of algebraic structures and ordered structures. The idea of a zero-divisor graph of a commutative ring $R$ with unity was introduced by Beck [5] as follows. Let $G$ be the simple graph whose vertices are the elements of $R$, with $x$ and $y$ adjacent if $xy = 0$. The graph $G$ is the *zero-divisor graph* of $R$. The chromatic number of a graph $G$ is denoted by $\chi(G)$. That is, $\chi(G)$ is the minimum number of colors in a coloring of the elements of $R$ such that adjacent elements receive different colors. If this number is not finite, write $\chi(G) = \infty$. A subset $C$ of $G$ is a *clique* if any two distinct vertices of $C$ are adjacent. The *clique number* of a graph $G$, written $\omega(G)$, is the maximum number of vertices in a clique in $G$. If the sizes of the cliques are not bounded, then $\omega(G) = \infty$. Always $\chi(G) \geq \omega(G)$. In [5], Beck conjectured that $\chi(G) = \omega(G)$ when $G$ is the zero-divisor graph of a commutative ring with unity, but Anderson and Naseer [4] gave an example of a commutative local ring $R$ with 32 elements for which $\chi(G) > \omega(G)$.

Many papers such as Anderson et al. [3], F. DeMeyer, T. McKenzie and K. Schneider [8], Maimani, Pournaki and Yassemi [23], Redmond [25], and Samei [26] investigated the interplay between algebraic properties of a structure and its graph-theoretic properties. The zero-divisor graphs of ordered structures were recently studied by Halaš and Jukl [11], Halaš and Länger [12], Joshi [13], Joshi et al. [15–17,20–22], Nimbhorkar et al. [24], etc.

In ring theory, the structure of a ring $R$ is closely related to the behavior of ideals. Hence Behboodi and Rakeei [6,7] introduced the concept of an annihilating-ideal graph $\mathbb{AG}(R)$ of a commutative ring $R$ with unity, where the vertex set $V(\mathbb{AG}(R))$ is the set of nonzero ideals with nonzero annihilator. That is, a nonzero ideal $I$ belongs to $V(\mathbb{AG}(R))$ if and only if there exists a nonzero ideal $J$ of $R$ such that $IJ = (0)$, and two distinct vertices $I$ and $J$ are adjacent if and only if $IJ = (0)$. In [7], Behboodi and Rakeei raised the following conjecture.

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**Conjecture 1.1.** *For every commutative ring R with unity,* $\chi(\mathbb{AG}(R)) = \omega(\mathbb{AG}(R))$.

It is interesting to observe that the set $Id(R)$ of all ideals of a commutative ring $R$ with unity forms a compactly generated 1-compact modular multiplicative lattice in which the product of two compact elements is compact (see Definition 1.3). Also, the annihilating-ideal graph of a commutative ring $R$ with unity is nothing but the multiplicative zero-divisor graph of the multiplicative lattice of all ideals of $R$, where the vertex set is the set of nonzero zero-divisors and vertices $a$ and $b$ are adjacent if and only if $ab = 0$. Hence when studying the annihilating-ideal graphs of a commutative ring with unity, a multiplicative lattice becomes an appropriate tool. This motivates us to define and study the multiplicative zero-divisor graph $\widetilde{\Gamma_I}(L)$ of a multiplicative lattice $L$ with respect to an ideal $I$ of $L$ (see Definitions 1.3 and 2.3). We say that a multiplicative lattice has the *Beck property* if the chromatic number and clique number of its multiplicative zero-divisor graph with respect to any ideal are equal. It is natural to ask the following question; an affirmative answer to it proves Conjecture 1.1. of Behboodi and Rakeei [7].

**Question 1.2.** *Does the Beck property hold for a given multiplicative lattice?*

In this paper, we introduce the multiplicative zero-divisor graph of a multiplicative lattice. We prove that the answer to Question 1.2 may be no when $L$ is a non-reduced multiplicative lattice but that it is always yes when $L$ is a reduced multiplicative lattice. The positive result extends the result of Behboodi and Rakeei [7] and Aalipour et al. [1].

We begin with necessary concepts and terminology.

**Definition 1.3.** A nonempty subset $I$ of a lattice $L$ is a *semi-ideal* if $x \leq a \in I$ implies $x \in I$. A semi-ideal $I$ of $L$ is an *ideal* if $a \vee b \in I$ whenever $a, b \in I$. An ideal (semi-ideal) $I$ of a lattice $L$ is a *proper* ideal (semi-ideal) of $L$ if $I \neq L$. A proper ideal (semi-ideal) $I$ is *prime* if $a \wedge b \in I$ implies $a \in I$ or $b \in I$, and it is *minimal* if it does not properly contain another prime ideal (prime semi-ideal). Dually, a nonempty subset $F$ of a lattice $L$ is a *semi-filter* if $x \geq y \in F$ implies $x \in F$. A semi-filter $F$ of $L$ is a *filter* if $a \wedge b \in F$ whenever $a, b \in F$. A filter $F$ of a lattice $L$ is a *proper* filter of $L$ if $F \neq L$. A proper filter $F$ is *prime* if $a \vee b \in F$ implies $a \in F$ or $b \in F$. A filter $F$ of a lattice $L$ is *maximal* if any filter containing it lies in $\{F, L\}$. For $a \in L$, let $(a] = \{x \in L\colon x \leq a\}$. The set $(a]$ is the *principal ideal generated by a*. Dually, we have the concept of a principal filter $[a)$ generated by $a$.

A lattice $L$ is *complete* if for any subset $S$ of $L$, we have $\bigvee S, \bigwedge S \in L$. The smallest element and the greatest element of a lattice $L$ are denoted by 0 and 1 respectively. A complete lattice $L$ is a *multiplicative lattice* if there exists a binary operation "$\cdot$" called *multiplication* on $L$ satisfying the following conditions:

(1) $a \cdot b = b \cdot a$, for $a, b \in L$,
(2) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, for $a, b, c \in L$,
(3) $a \cdot (\bigvee_\alpha b_\alpha) = \bigvee_\alpha (a \cdot b_\alpha)$, for $a, b_\alpha \in L$,
(4) $a \cdot b \leq a \wedge b$ for $a, b \in L$,
(5) $a \cdot 1 = a$, for $a \in L$.

An element $c$ of a complete lattice $L$ is *compact* if $c \leq \bigvee_\alpha a_\alpha$ implies $c \leq \bigvee_{i=1}^n a_{\alpha_i}$, where $n \in \mathbb{Z}^+$. The set of all compact elements of a lattice $L$ is denoted by $L_*$. A lattice $L$ is *compactly generated* or *algebraic* if for every $x \in L$, there exist $x_\alpha \in L_*$ for $\alpha \in \Lambda$ (the index set) such that $x = \bigvee_\alpha x_\alpha$, that is, every element is a join of compact elements. A multiplicative lattice $L$ is 1-*compact* if 1 is a compact element of $L$. A multiplicative lattice $L$ is *compact* if every element is a compact element.

An element $p$ of a multiplicative lattice $L$ with $p \neq 1$ is *prime* if $a \cdot b \leq p$ implies $a \leq p$ or $b \leq p$. It is not difficult to prove that an element $p$ (with $p \neq 1$) of a 1-compact, compactly generated lattice $L$ is *prime* if $a \cdot b \leq p$ for $a, b \in L_*$ implies $a \leq p$ or $b \leq p$.

A nonempty subset $S$ of $L_*$ in a 1-compact, compactly generated lattice $L$ is *multiplicatively closed* if $s_1 \cdot s_2 \in S$ whenever $s_1, s_2 \in S$.

In a multiplicative lattice $L$, an element $a \in L$ is *nilpotent* if $a^n = 0$ for some $n \in \mathbb{Z}^+$, and $L$ is *reduced* if the only nilpotent element is 0. For an element $a$ of a multiplicative lattice, we define $a^* = \bigvee\{x \in L\colon a^n \cdot x = 0 \text{ for some positive integer } n\}$. If $L$ is reduced, then $a^* = \bigvee\{x \in L\colon x \cdot a = 0\}$.

A lattice $L$ with 0 is 0-*distributive* if $a \wedge b = 0 = a \wedge c$ implies $a \wedge (b \vee c) = 0$; see Varlet [27]. The concept of 0-distributive poset can be found in [18,19].

Let $P$ be a poset and $A \subseteq P$. Let $A^u = \{x \in P\colon x \geq a \text{ for every } a \in A\}$. Dually, we have the set $A^\ell$. By $A^{u\ell}$, we mean $\{A^u\}^\ell$. A subset $I$ of a poset $P$ is an *ideal* of $P$ if $a, b \in I$ implies $\{a, b\}^{u\ell} \subseteq I$.

## 2. Multiplicative zero-divisor graph of a multiplicative lattice

Joshi [13] introduced the zero-divisor graph of a poset with respect to an ideal $I$. We recall this definition, when the poset is a lattice.

**Definition 2.1.** Let $I$ be an ideal of a lattice $L$. We associate to $I$ an undirected simple graph $\Gamma_I(L)$, called the *zero-divisor graph of L with respect to I*, in which the set of vertices is $\{x \notin I\colon x \wedge y \in I \text{ for some } y \notin I\}$ and distinct vertices $a, b$ are adjacent if and only if $a \wedge b \in I$. Whenever $I = (0]$, we denote $\Gamma_I(L)$ by $\Gamma(L)$.

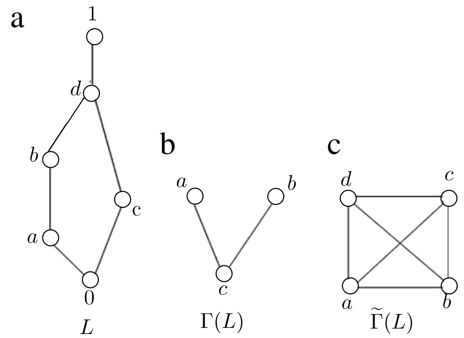


**Fig. 1.** The zero-divisor and multiplicative zero-divisor graph.

**Table 1**
Multiplication table.

| $\bullet$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $(a \vee c)$ | $(a \vee d)$ | $(b \vee e)$ | $(c \vee e)$ | $(b \vee d)$ | $t$ | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | $f$ | 0 | $f$ | $f$ | 0 | 0 | $f$ | $f$ | 0 | $f$ | $f$ | $f$ | $a$ |
| $b$ | 0 | 0 | $f$ | 0 | $f$ | $f$ | 0 | 0 | $f$ | $f$ | $f$ | $f$ | $f$ | $b$ |
| $c$ | 0 | $f$ | 0 | $f$ | 0 | $f$ | 0 | $f$ | $f$ | $f$ | $f$ | 0 | $f$ | $c$ |
| $d$ | 0 | $f$ | $f$ | 0 | $f$ | 0 | 0 | $f$ | $f$ | $f$ | 0 | $f$ | $f$ | $d$ |
| $e$ | 0 | 0 | $f$ | $f$ | 0 | $f$ | 0 | $f$ | 0 | $f$ | $f$ | $f$ | $f$ | $e$ |
| $f$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $f$ |
| $(a \vee c)$ | 0 | $f$ | 0 | $f$ | $f$ | $f$ | 0 | $f$ | $f$ | $f$ | $f$ | $f$ | $f$ | $(a \vee c)$ |
| $(a \vee d)$ | 0 | $f$ | $f$ | $f$ | $f$ | 0 | 0 | $f$ | $f$ | $f$ | $f$ | $f$ | $f$ | $(a \vee d)$ |
| $(b \vee e)$ | 0 | 0 | $f$ | $f$ | $f$ | $f$ | 0 | $f$ | $f$ | $f$ | $f$ | $f$ | $f$ | $(b \vee e)$ |
| $(c \vee e)$ | 0 | $f$ | $f$ | $f$ | 0 | $f$ | 0 | $f$ | $f$ | $f$ | $f$ | $f$ | $f$ | $(c \vee e)$ |
| $(b \vee d)$ | 0 | $f$ | $f$ | 0 | $f$ | $f$ | 0 | $f$ | $f$ | $f$ | $f$ | $f$ | $f$ | $(b \vee d)$ |
| $t$ | 0 | $f$ | $f$ | $f$ | $f$ | $f$ | 0 | $f$ | $f$ | $f$ | $f$ | $f$ | $f$ | t |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $(a \vee c)$ | $(a \vee d)$ | $(b \vee e)$ | $(c \vee e)$ | $(b \vee d)$ | $t$ | 1 |

We illustrate this concept with an example.

**Example 2.2.** The lattice $L$ and its zero-divisor graph $\Gamma(L)$ (in the sense of Joshi [13]) are shown below.

Next, we introduce the concept of a multiplicative zero-divisor graph $\widetilde{\Gamma}_I(L)$ of a multiplicative lattice $L$ and illustrate it with an example.

**Definition 2.3.** Let $I$ be an ideal (semi-ideal) of a multiplicative lattice $L$. We associate to $I$ an undirected simple graph $\widetilde{\Gamma}_I(L)$, called the *multiplicative zero-divisor graph of* $L$ *with respect to* $I$ and denoted by $\widetilde{\Gamma}_I(L)$, in which the set of vertices is $\{x \notin I \colon x \cdot y \in I$ for some $y \notin I\}$, and two distinct vertices $a, b$ are adjacent if and only if $a \cdot b \in I$. When $I = \{0\}$, we denote $\widetilde{\Gamma}_I(L)$ by $\widetilde{\Gamma}(L)$.

**Example 2.4.** Consider the same lattice $L$ shown in Fig. 1(a) with the trivial multiplication $x \cdot y = 0 = y \cdot x$ for every $x \neq 1 \neq y$ and $x \cdot 1 = x = 1 \cdot x$ for every $x \in L$. It is easy to see that $L$ is a multiplicative lattice. Further, its multiplicative zero-divisor graph $\widetilde{\Gamma}(L)$ is shown in Fig. 1(c). It is interesting to note that if the greatest element 1 is completely join-irreducible (that is, $1 = \bigvee x_i \Rightarrow 1 = x_i$ for some $i$), then any lattice with this trivial multiplication is a multiplicative lattice.

For undefined concepts in lattices and graphs, see Grätzer [9] and West [28] respectively.

It is known that $\chi(\Gamma_I(P)) = \omega(\Gamma_I(P))$ for the zero-divisor graph of a poset $P$ (with 0) with respect to an ideal $I$ of $P$, see [13, Theorem 2.9] (when $I = \{0\}$); see also [11, Theorem 2.13]. Hence it is natural to ask Question 1.2.

We answer Question 1.2 negatively in the following example.

**Example 2.5.** Consider the lattice $L$ depicted in Fig. 2(a). Define multiplication on $L$ as given in Table 1. It is not difficult to prove that $L$ is a multiplicative lattice. Moreover, $f^2 = 0$ for $f \neq 0$, so $L$ is non-reduced. Now, consider the multiplicative zero-divisor graph $\widetilde{\Gamma}(L)$ of $L$ depicted in Fig. 2(b). Note that $4 = \chi(\widetilde{\Gamma}(L)) > \omega(\widetilde{\Gamma}(L)) = 3$. Thus the Beck property does not hold for all multiplicative lattices.

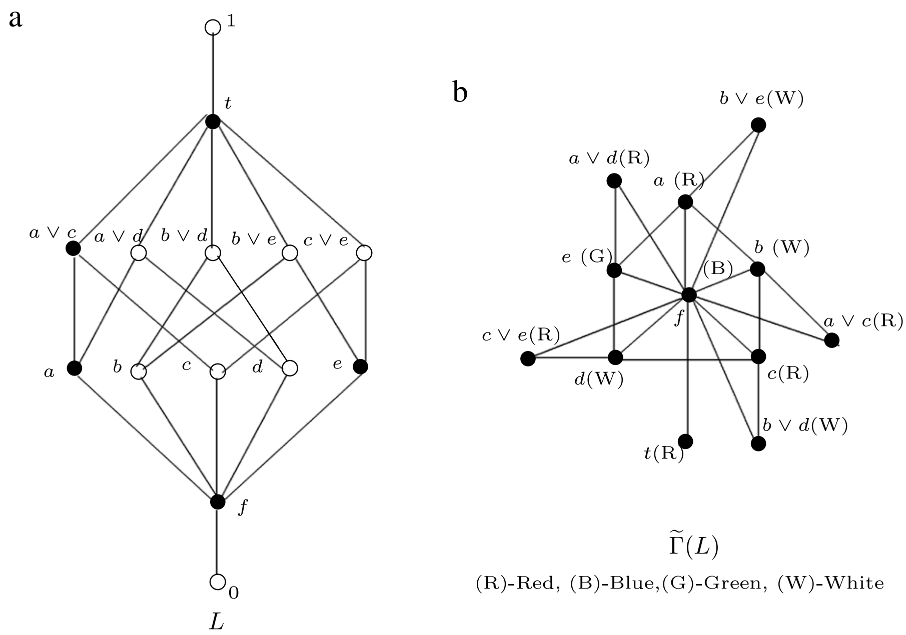


**Fig. 2.** Multiplicative lattice for which the Beck property does not hold.

**Remark 2.6.** When $R$ is a commutative ring with unity, it is well known that the ideal lattice $Id(R)$ of $R$ is a 1-compact, compactly generated modular multiplicative lattice; see Anderson [2]. Further, it is easy to observe that if $R$ is reduced, then $Id(R)$ is a reduced multiplicative lattice. The lattice depicted in Fig. 2(a) is a non-modular lattice (the heavy black elements form a non-modular sublattice), and hence it cannot be an ideal lattice of any commutative ring with unity. Therefore Conjecture 1.1 remains open even though we know that the Beck property does not hold for all non-reduced multiplicative lattices. We have a more pleasant situation for reduced multiplicative lattices. For this, we need Theorem 2.9 of [13]. Note that the notion of prime semi-ideals mentioned in [13] coincides with the corresponding notions in lattices. Hence we quote essentially Theorem 2.9 of [13], for the special case of lattices.

**Theorem 2.7** (*Joshi [13]*). *Let $L$ be a lattice with 0. If $\omega(\Gamma(L))$ is finite, then the chromatic number and clique number of $\Gamma(L)$ both equal the number of minimal prime semi-ideals in $L$.*

**Lemma 2.8.** *If $L$ is a reduced multiplicative lattice, then $L$ is 0-distributive.*

**Proof.** Let $a \wedge b = 0 = a \wedge c$ for $a, b, c \in L$. Since $a \cdot b \leq a \wedge b$ and $L$ is a multiplicative lattice, we have $a \cdot (b \vee c) = 0$. Furthermore, since $L$ is reduced, $a \cdot b = 0$ implies $a \wedge b = 0$. This together with $a \cdot (b \vee c) = 0$ proves that $L$ is 0-distributive. □

It was proved in Joshi and Mundlik [18] that every prime semi-ideal in a 0-distributive poset is a prime ideal of a 0-distributive poset. For the sake of completeness, we provide the proof of this in the following result, which is essential in proving the Beck property for reduced multiplicative lattices.

**Theorem 2.9.** *Let $L$ be a reduced multiplicative lattice. If the clique number of $\widetilde{\Gamma}(L)$ is finite, then the chromatic number and clique number of $\widetilde{\Gamma}(L)$ both equal the number of minimal prime ideals in $L$.*

**Proof.** If $L$ is a reduced multiplicative lattice, then $a \cdot b = 0$ if and only if $a \wedge b = 0$ for $a, b \in L$. By Lemma 2.8, $L$ is 0-distributive.

Next, we prove that every minimal prime semi-ideal of $L$ is a minimal prime ideal of $L$. Let $I$ be a minimal prime semi-ideal of $L$. To prove that $I$ is an ideal, it is enough to show that if $a, b \in I$, then $a \vee b \in I$. Let $a, b \in I$. Since $I$ is a minimal prime semi-ideal of $L$, it follows that $L \setminus I$ is a maximal filter of $L$. Also, since $a, b \notin L \setminus I$, we have $[a) \vee (L \setminus I) = [b) \vee (L \setminus I) = L = [0)$. Hence there exists $t \in L \setminus I$ such that $t \wedge a = 0 = t \wedge b$. By Lemma 2.8, we have $t \wedge (a \vee b) = 0$. This proves $a \vee b \in I$, since otherwise $0 = t \wedge (a \vee b) \in L \setminus I$, a contradiction to the maximality of $L \setminus I$. This proves that every minimal prime semi-ideal is a minimal prime ideal.

Since $a \cdot b = 0$ if and only if $a \wedge b = 0$ for $a, b \in L$, the zero-divisor graph $\Gamma(L)$ of the lattice $L$ is isomorphic to the multiplicative zero-divisor graph $\widetilde{\Gamma}(L)$ of the reduced multiplicative lattice $L$. Since $\omega(\widetilde{\Gamma}(L)) < \infty$, we have $\omega(\Gamma(L)) < \infty$. By using Theorem 2.7 and the fact that every lattice is a reduced multiplicative lattice with multiplication being the meet operation, we have $\chi(\widetilde{\Gamma}(L)) = \omega(\widetilde{\Gamma}(L)) = n$, where $n$ is the number of minimal prime ideals of $L$. □

**Remark 2.10.** It is obvious that the prime ideals in a commutative ring $R$ with unity are nothing but the prime elements of the multiplicative lattice $Id(R)$. Note that the annihilating-ideal graph $\mathbb{AG}(R)$ of a commutative ring $R$ with unity is nothing but the multiplicative zero-divisor graph of a multiplicative lattice $Id(R)$ of all ideals of $R$. Hence Theorem 2.9 extends Corollary 2.11 of Behboodi and Rakeei [7], but not completely Theorem 8 of Aalipour et al. [1]. In order to extend Theorem 8 of Aalipour et al. [1], we have to prove that $\chi(\widetilde{\Gamma}(L)) = \omega(\widetilde{\Gamma}(L)) = n$, where $L$ is a reduced multiplicative lattice and $n$ is its number of minimal prime elements. We achieve this result in the sequel.

First, we provide an example of a reduced multiplicative lattice that has no prime element. A reduced multiplicative lattice always has a prime ideal but need not have a prime element. To see this, let $\mathbb{N}$ be the set of natural numbers. Let $L = \{X \subseteq \mathbb{N}\colon |X| < \infty\} \cup \{\mathbb{N}\}$. It is easy to see that $L$ is a lattice under set inclusion. In fact, $L$ is a reduced multiplicative lattice with multiplication being the meet operation. Let $\{n\}^{\perp} = \{A \subseteq \mathbb{N}\colon A \cap \{n\} = \emptyset \text{ and } |A| < \infty\}$. One can prove that $\{n\}^{\perp}$ is a minimal prime ideal of $L$ for every $n \in \mathbb{N}$. However, $L$ does not contain any prime element.

**Lemma 2.11.** *Let $L$ be a reduced, 1-compact, compactly generated lattice. For $x \in L$, if $x^*$ is maximal among $\{a^*\colon a \in L, a^* \neq 1\}$, then $x^*$ is a prime element of $L$.*

**Proof.** If $a \cdot b \le x^*$ and $a \nleq x^*$, then $(x \cdot a \cdot b) = 0$. Let $(0 \neq)\, y \le (x \cdot a)$. Since $(b \cdot y) \le (x \cdot a \cdot b) = 0$, we have $b \le y^*$. Since $(0 \neq)\, y \le x$ and $x^*$ is maximal, $y^* = x^*$, and hence $b \le x^*$. This proves that $x^*$ is prime. □

**Lemma 2.12.** *Let $L$ be a reduced, 1-compact, compactly generated lattice. If $x^*$ and $y^*$ are distinct prime elements of $L$, then $x \cdot y = 0$.*

**Proof.** Assume to the contrary that $x \cdot y \neq 0$, that is, $x \nleq y^*$ and $y \nleq x^*$. Let $t$ be a compact element such that $t \le x^*$. Note that $x \cdot t = 0$ and $x \cdot t \le y^*$. Since $L$ is compactly generated and every compact element $t \le x^*$ is also below $y^*$, we have $t \le y^*$ and hence $x^* \le y^*$. Similarly, we can show $y^* \le x^*$. Hence $y^* = x^*$, a contradiction. □

**Lemma 2.13.** *If $L$ is a reduced, 1-compact, compactly generated lattice such that $\omega(\widetilde{\Gamma}(L)) < \infty$, then the set $\{x^*\colon x \in L, x \neq 0\}$ satisfies the ascending chain condition.*

**Proof.** Suppose $a_1^* < a_2^* < a_3^* < a_4^* \cdots$. Since $L$ is compactly generated and $a_{j-1}^* < a_j^*$, we have $x_j \in L_*$ such that $x_j \le a_j^*$ and $x_j \nleq a_{j-1}^*$, for $j \ge 2$. If we let $y_j = (x_j \cdot a_{j-1}), j \ge 2$, then $y_j \neq 0$. For $i < j$, we have $x_i \le a_i^* \le a_{j-1}^*$. Thus $(x_i \cdot a_{j-1}) = 0$; consequently, $(y_i \cdot y_j) = 0$ for all $i \neq j$. Thus the set $\{y_j\colon j \ge 2\}$ is an infinite clique, a contradiction. □

**Lemma 2.14.** *Let $L$ be a reduced, 1-compact, compactly generated lattice. If $\omega(\widetilde{\Gamma}(L))$ is finite, then the set of all distinct maximal annihilator elements of $L$ is finite.*

**Proof.** Let $A = \{x_i^*\colon x_i^* \text{ is maximal}\}$ be the set of all maximal annihilator elements of $L$. Clearly, $x_i \neq 0$ for all $i$ and $x_i^* \neq x_j^*$ whenever $i \neq j$. By Lemma 2.11, all the elements of $A$ are prime. By Lemma 2.12, $x_i \cdot x_j = 0$ for all $i \neq j$. This shows $\omega(\widetilde{\Gamma}(L)) \ge |A|$, which according to $\omega(\widetilde{\Gamma}(L)) < \infty$ yields the finiteness of $A$. □

**Lemma 2.15.** *Let $L$ be a reduced, 1-compact, compactly generated lattice with $L_*$ multiplicatively closed set. If $\omega(\widetilde{\Gamma}(L))$ is finite, then 0 is the meet of a finite number of minimal prime elements of $L$.*

**Proof.** According to Lemma 2.14, let $\{x_i^*\colon 1 \le i \le n\}$ be the set of all maximal annihilator elements of $L$. By Lemma 2.11, all these elements are prime. Further, due to Lemma 2.12, $x_i \cdot x_j = 0$ for all $i \neq j$. Assume that there exists a nonzero element $a$ such that $a \le \bigwedge_{i=1}^{n} x_i^*$. Since $a \cdot x_i = 0$ for all $i$, we have $x_i \le a^*$ for all $i$. However, Lemma 2.13 guarantees $a^* \le x_i^*$ for some $i$. However, this yields $x_i \le a^* \le x_i^*$, that is, $x_i^2 = 0$, which contradicts the reducibility of $L$. Thus $0 = \bigwedge_{i=1}^{n} x_i^*$. We denote $x_i^*$ by $p_i$.

Next, we show that $p_i$ are minimal prime elements of $L$. Since $p_i$ are assumed to be maximal annihilator elements, we may suppose for $i \neq j$ none of $p_i$ contains $p_j$. Indeed, if there exists $j$ such that $p_j$ is not minimal, then there exists a minimal prime element $q$ with $q < p_j$. Now, since $\bigwedge_{i=1}^{n} x_i^* \le q$ and $q$ is prime, we have $x_i^* \le q$ for some $i$. This yields $x_i^* = p_i \le q \le p_j$, a contradiction. Thus $\bigwedge_{i=1}^{n} p_i = 0$. □

If the smallest element 0 of a multiplicative lattice $L$ is a meet of finite number of minimal prime elements, say $n$, then 0 is a *$n$-prime element*. More details about $n$-prime elements can be found in Joshi and Ballal [14]. This concept is analogous to the concept of $n$-prime semi-ideals introduced by Halaš [10].

**Lemma 2.16.** *Let $L$ be a reduced, 1-compact, compactly generated lattice with multiplicatively closed set $L_*$. If $\omega(\widetilde{\Gamma}(L))$ is finite, then every minimal prime element $p$ of $L$ is of the form $x^*$ for some $x \in L$.*

**Proof.** Let $p$ be a minimal prime element of $L$, and let $A = \{x^*\colon x \nleq p\}$. For $x \nleq p$, we have $x^* \le p$. By Lemma 2.13, there are maximal annihilator elements in $A$. In fact, we prove that there is a greatest one. Let $y_1^*, y_2^*$ be two maximal elements of $A$. We have $y_1 \cdot y_2 \nleq p$, since $p$ is prime and $y_1, y_2 \nleq p$. Thus there exists a nonzero element $y$ with $y \nleq p$ such that $y = y_1 \cdot y_2$. Clearly, $y_1^*, y_2^* \le y^*$. Since $y_1^*$ and $y_2^*$ are both maximal in $A$, we conclude that $y_1^* = y_2^* = y^*$. This shows that $A$ has a greatest element; call it $z^*$.

By Lemma 2.11, $z^*$ is prime. We prove $z^* \leq p$. If not, then there is a compact element $g \leq z^*$ such that $g \nleq p$. Now $z \leq z^{**} \leq g^* \in A$. Hence we have $z \leq g^* \leq z^*$, contradicting that $L$ is reduced. Thus $z^* \leq p$ and $p$ is a minimal prime element of $L$, so $p = z^*$. □

Finally, we prove the Beck property for a particular class of multiplicative lattices, namely reduced multiplicative lattices.

**Theorem 2.17.** *Let $L$ be a reduced, 1-compact, compactly generated lattice with multiplicatively closed set $L_*$. If $\omega(\widetilde{\Gamma}(L))$ is finite, then the chromatic number and clique number of $\widetilde{\Gamma}(L)$ both equal the number of minimal prime ideals in $L$.*

**Proof.** By Lemma 2.15, we have $0 = \bigwedge_{i=1}^{n} p_i$ whenever $p_1, \ldots, p_n$ are minimal prime elements of $L$. By Lemma 2.16, $p_i = x_i^*$ for some $x_i \neq 0$; hence $0 = \bigwedge_{i=1}^{n} x_i^*$. By Lemma 2.12, $\{x_i\colon 1 \leq i \leq n\}$ is a clique in $L$ and thus $\omega(L) \geq n$. Define a coloring of $L$ by $f(x) = \min\{i\colon x \nleq p_i\}$. If $x$ and $y$ are adjacent vertices, then $x \cdot y = 0$. If $f(x) = k+1$, then $x \leq p_i$ for $1 \leq i \leq k$ and $x \nleq p_{k+1}$. We conclude that $y \leq p_{k+1}$, since $x \cdot y = 0 \leq p_{k+1}$ and $x \nleq p_{k+1}$. This shows that $f(y) \neq k+1$, and thus $f(x) \neq f(y)$. Hence $f$ is a proper coloring of $L$. This yields $\chi(\widetilde{\Gamma}(L)) \leq n$, and finally $n \leq \omega(\widetilde{\Gamma}(L)) \leq \chi(\widetilde{\Gamma}(L)) \leq n$. In conclusion, we have $\chi(\widetilde{\Gamma}(L)) = \omega(\widetilde{\Gamma}(L)) = n$. □

**Corollary 2.18** (*Aalipour et al. [1]*). *If $R$ is a reduced commutative ring with unity such that $\omega(\mathbb{AG}(R)) < \infty$, then $\chi(\mathbb{AG}(R)) = \omega(\mathbb{AG}(R)) = |\mathrm{Min}(R)|$, where $\mathrm{Min}(R)$ is the set of all minimal prime ideals of $R$.*


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